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## **Abstract**

For any abstract bargaining problem a non-cooperative one stage strategic game is constructed whose unique dominant strategies Nash equilibrium implements the Nash solution of the bargaining problem.

## **Keywords**

Nash program, implementation, Nash bargaining solution

## **JEL-Classifications**

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#### **Comments**

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# 1. Introduction

After his seminal paper Nash (1950) where he introduced his bargaining solution Nash (1953) also suggested to base cooperative axiomatic solutions on equilibria of non-cooperative strategic games. This idea is known as the "Nash program" in the game theoretic literature. Thorough discussions of the Nash program are contained in Binmore and Dasgupta (1987) and in Osborne and Rubinstein (1990). Extensive treatments of axiomatic bargaining theory are Roth (1979) and Thomson and Lensberg (1989).

Nash (1953) himself started with a contribution to the Nash program by introducing what is called today the *simple Nash demand* game. The continuum of equilibria of this game, which coincides with the efficient boundary of the bargaining problem is not suited to single out the Nash solution. A modification, also due to Nash, with informational trembling which anticipated features of Selten's (1975) perfectness resulted in an approximate implementation of the Nash solution. The most prominent later contribution to the Nash program is Rubinstein's (1982) alternate bargaining model in which the Nash solution turns out to be implementable as a limit of perfect equilibria in a sequence of certain multi-stage games. As Binmore (1987) remarks "none of the non-cooperative bargaining models who have been studied implement the Nash bargaining solution *exactly*. In each case the implementation is *approximate* (or exact only in the limit)."

An *exact* implementation of the Nash solution by a subgame perfect equilibrium of an extensive game has been provided recently by Howard (1992). His game has been reproduced in Osborne and Rubinstein (1994).

In the present paper we suggest a different non-cooperative approach to the bargaining problem, which is based on Shapley's (1969) idea of endogenously determined transfer rates between players' utilities. Here a bargaining problem is viewed as a specific Arrow-Debreu economy with production and private ownership (cf. Trockel (1994)). It turns out that the unique (stable) Walrasian equilibrium of this economy coincides with the Nash solution of the bargaining problem. It is this competitive approach to the bargaining problem which then suggests a modification of Nash's simple demand game where the payoff function is derived from demand and supply in the economy. The unique Nash equilibrium, which is even an equilibrium in dominant strategies, implements *exactly* the Nash bargaining solution.

## 2. Motivation and Description of the Game

To introduce and discuss our game we first recall how Nash described a bargaining situation.

A two person bargaining problem is a pair  $(S, d)$  where the *feasible set*  $S$  is a convex subset of  $\mathbb{R}^2$  and  $d$  is an element of  $S$ . The point  $d$  is interpreted as the *status quo* or *disagreement point*, while each point in  $S$  is interpreted as a feasible pair of utility levels of the two players. Implicitly this interpretation is based on the idea of some underlying unspecified economic or social model whose different states evaluated by the two players' cardinal utility functions result in the elements of  $S$ . In the most frequent interpretation one thinks of von Neumann–Morgenstern utility functions in a lottery setup.

It was Nash (1953) himself who tried to support his axiomatic bargaining solution by a strategic non-cooperative model. In his static model both players announce "demands" of utility levels for themselves. If these are compatible, i.e. if the pair of announced demands builds an element of  $S$  each player receives the amount of utility he demanded. Otherwise the disagreement point results. The continuum of Nash equilibria of this *simple demand game* contains all efficient elements of  $S$  as well as the disagreement point.

Nash extended his analysis to a more elaborate version where he considered *perturbed demand games*. Anticipating features of Selten's (1975) perfectness Nash singled out those equilibria which are robust with respect to specific perturbations reflecting some uncertainty about the outcome. The set of those equilibria of the perturbed games generating agreement with positive probability is the smaller the smaller the perturbation is and converges to the Nash solution of  $(S, d)$ .

In the present paper we consider quite a different modification of Nash's simple demand game. We think of  $S$  as of a technology set representing all feasible production plans where pairs of utility levels for the two players are the two commodities which may be produced jointly without any input. (This assumption might appear confusing but it can be easily removed by introducing a third input commodity without direct utility to the players). Imagine now that all points  $y$  in the efficient boundary  $\partial S$  are evaluated by their associated efficiency prices  $p(y)$ , which are assumed to be normalized by  $p(y) \cdot y = 1$ . For any efficient joint production plan  $y \in \partial S$  half of its value  $\frac{1}{2} p(y) \cdot y$  is

made available to each player. He may choose now his *demand*, i.e. the optimal bundle in his budget set. Considered as an agent of an economy each player is interested only in one of the two commodities, namely his own utility. Accordingly both players' preferences are described by indifference lines parallel to their axes. Their optima are the corner solutions  $(\frac{1}{2p_1(y)}, 0)$  for player 1 and  $(0, \frac{1}{2p_2(y)})$  for player 2.

The possible conflict between the proposed supply vector  $y \in \partial S$  and the resulting aggregate demand  $(\frac{1}{2p_1(y)}, \frac{1}{2p_2(y)})$  determines our equilibrium approach and is the key to the subsequent strategic approach.

### 3. Model and Results

Let  $S \subset \mathbb{R}^2$  be compact, strictly convex, comprehensive with respect to  $\mathbb{R}_+^2$  (i.e.  $x \in S \Rightarrow \{x' \mid 0 \leq x' \leq x\} \subset S$ ) and  $d \in S$ .

The pair  $(S, d)$  is called a *two person bargaining problem* if there is some  $x \in S$  such that  $x \succ d$ .  $[x \succ x' : \Leftrightarrow \forall i = 1, 2: x_i > x'_i; x \succeq x' : \Leftrightarrow \forall i = 1, 2: x_i \geq x'_i; x \succ x' : \Leftrightarrow (x \succeq x' \text{ and } x \neq x')]$

Having cardinality in mind we may and do assume  $d = 0$  without loss of generality.

Let  $\mathcal{B}$  denote the set of all two person bargaining problems  $(S, 0)$ .

A *bargaining solution* is a map

$$\varphi: \mathcal{B} \rightarrow \mathbb{R}^2: (S, 0) \mapsto \varphi(S, 0) \in S.$$

The *Nash bargaining solution*  $\varphi_N$ , originally defined by Nash (1950) through some axioms, has (also by Nash) been characterized by

$$\varphi_N(S, 0) \mapsto \operatorname{argmax}_{x \in S} x_1 \cdot x_2,$$

i.e. by associating with each  $(S, 0)$  that point where the so called "Nash product" becomes maximal.

Next we consider any bargaining problem  $(S, 0)$  as a specific *Arrow-Debreu economy* with production and private ownership.

Given  $(S, 0)$  consider an economy  $\mathcal{E}_S$  defined as follows:

$\mathcal{E}_S = (\lambda_1, \lambda_2; e_1, e_2; \vartheta_1, \vartheta_2; Y)$  with production possibility set  $Y \subset \mathbb{R}^2$ , preferences  $\lambda_i$ , endowment vectors  $e_i$  and ownership shares in production  $\vartheta_i$ ,  $i = 1, 2$ . Specifically we define  $Y := S$ ,  $e_1 = e_2 = (0, 0)$ ,  $\vartheta_1 = \vartheta_2 = \frac{1}{2}$  and  $\lambda_i$  by:  $x \succ_i x' : \Leftrightarrow x_i > x'_i$ ,  $i = 1, 2$ . The interpretation of this economy is as follows:

Two agents jointly own a firm which can produce joint utilities for the two agents. Each agent is interested only in one of the two commodities, namely his utility. The only income results from shares in profit from production. The confusing property of production without inputs could be easily removed by introduction of a third (input) commodity, a fixed amount of which is required for production, consumption of which, however, does not give utilities to the players.

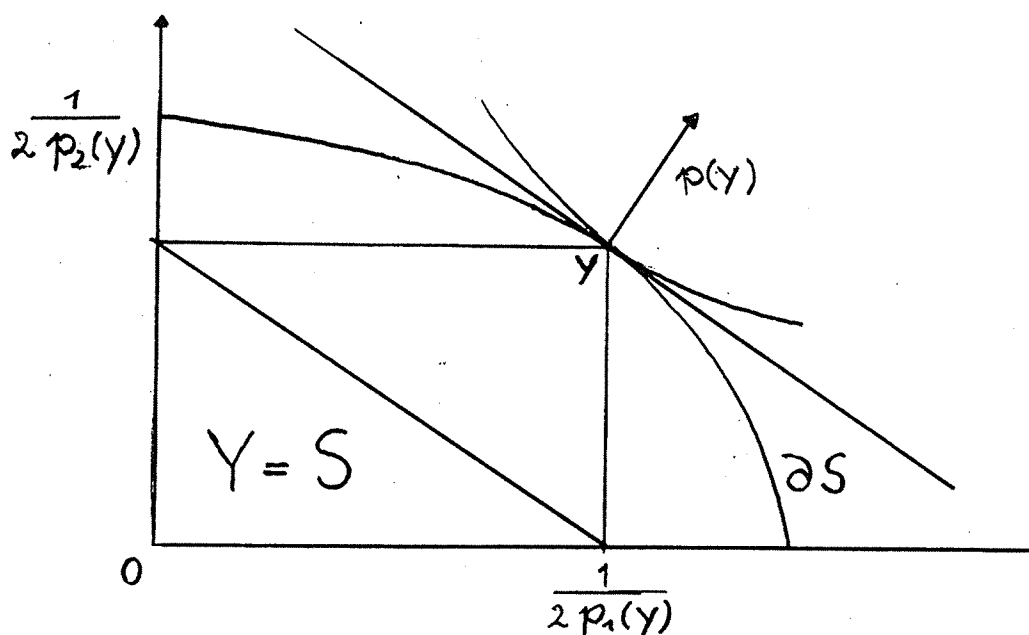


Figure 1

An *equilibrium* of the economy is characterized as follows:

Take a point  $y$  in the efficient boundary  $\partial Y$  and evaluate it by its *efficiency price* (a normal to  $\partial Y$  at  $y$ ). Make half of the resulting value available as *income* to each player. Determine both players' *individual demands* and add them up to get the *aggregate demand*. This aggregate demand has to coincide with  $y$ .

Without loss of generality we assume that all efficiency prices  $p(y)$  for  $y \in \partial Y$  are normalized by  $p(y) \cdot y = 1$ . This normalization which has been motivated in Debreu (1954) will be notationally convenient.

Note that for any efficient production plan  $y \in \partial Y$  both players have the budget set  $B(y) := \{x \in \mathbb{R}_+^2 \mid p(y) \cdot x \leq \frac{1}{2} p(y) \cdot y = \frac{1}{2}\}$ . Given their specific preferences the individual demand vectors are  $(\frac{1}{2p_1(y)}, 0)$  and  $(0, \frac{1}{2p_2(y)})$ , for player 1 and 2 respectively. The resulting aggregate demand  $(\frac{1}{2p_1(y)}, \frac{1}{2p_2(y)})$  can be interpreted as the demand of a hypothetical *representative consumer* whose preference is represented by the Cobb–Douglas utility function  $(x_1, x_2) \mapsto x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ .

Therefore the equilibria of  $\mathcal{E}_S$  coincide with the optimal choices of this representative agent on  $Y$ . Due to convexity of  $Y$  and strict quasi-concavity of the Cobb–Douglas utility function the representative consumer has a unique optimum. Hence the economy  $\mathcal{E}_S$  has a unique competitive equilibrium. But this unique optimum is just the point which maximizes the Nash product  $x_1 \cdot x_2$  on  $S = Y$ .

So we have proved the following result which for  $n$ -person bargaining games has been stated in Trockel (1994).

**Proposition 1:**      Given a two person bargaining problem  $(S, 0)$ . The economy  $\mathcal{E}_S$  has a unique Walrasian equilibrium. The equilibrium production plan and the aggregate equilibrium consumption coincide with the Nash solution  $N = \varphi_N(S, 0)$  of  $(S, 0)$ .

Next we define a one-stage non-cooperative game in strategic form whose payoff functions are derived from the individual demand functions in  $\mathcal{E}_S$ . To simplify the arguments we assume now in addition that  $\partial S$  is smooth, i.e. at each point  $y \in \partial S$  there is a unique (normalized)  $p(y)$ .

For any  $y \in \partial S$  define  $z_i(y) := \min(y_i, \frac{1}{2p_i(y)})$ ,  $i = 1, 2$ .

We define the game  $\Gamma_S$  associated with a bargaining problem  $(S, 0)$  as follows:

The strategy sets  $S_i$  are the projections of  $S$  to the  $i$ -th axis, i.e.  $S_i = \text{proj}_i S$ ,  $i = 1, 2$ .

Next observe that any  $x = (x_1, x_2) \in S_1 \times S_2$  defines two points  $y^1(x), y^2(x) \in \partial S$  by  $y^1(x) = x_1$  and  $y^2(x) = x_2$  which coincide if and only if  $x \in \partial S$ .

The payoff functions  $\zeta_i$  for the game  $\Gamma_S$  are defined by:

$$\zeta_i(x) := z_i(y^i(x))$$

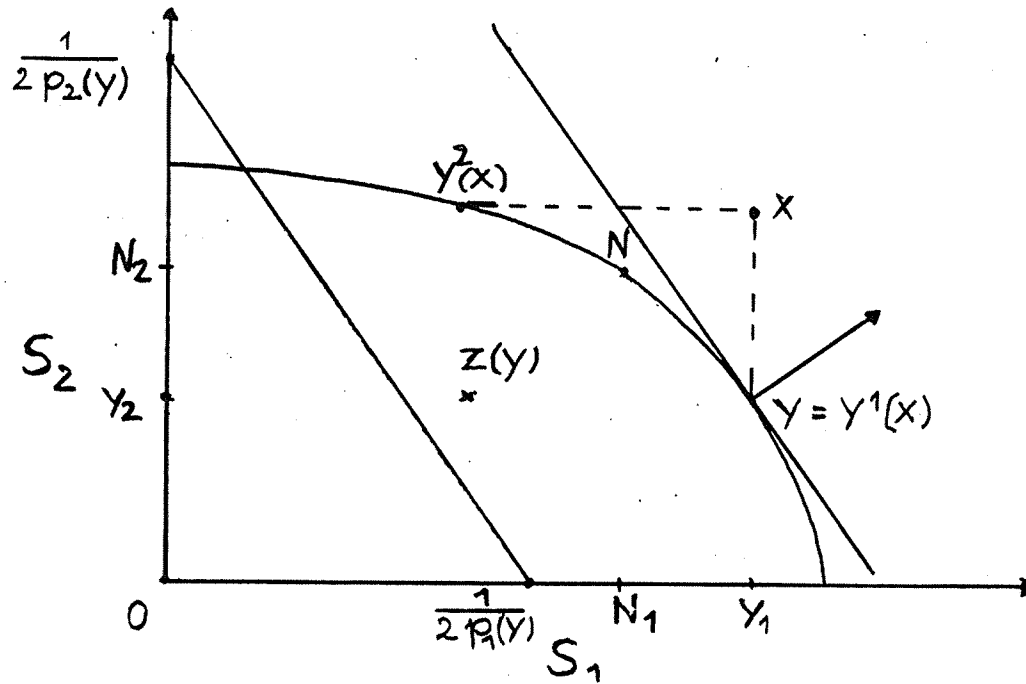


Figure 2

So we get  $\Gamma_S = (S_1, S_2; \zeta_1, \zeta_2)$ .

Now we can state our result.

**Proposition 2:** Given a two person bargaining problem  $(S, 0)$ . The game  $\Gamma_S$  has a unique Nash equilibrium. This is even an equilibrium in dominant strategies and coincides with the Nash solution  $N = \varphi_N(S, 0)$ .

**Proof:**

First observe that for  $N \in \partial S$  we have  $y^i(N) = N$ ,  $i = 1, 2$  and therefore  $\zeta_i(N) = z_i(N) = N_i = \frac{1}{2p_i(N)}$  by Proposition 1.

Next observe that  $\partial S$  is the graph of two strictly decreasing, strictly concave functions  $f_1$  and  $f_2$  of  $y_1$  and of  $y_2$ , respectively. Accordingly the functions  $y_i \mapsto \frac{1}{2p_i(y_i, f_i(y_i))}$ ,  $i = 1, 2$  are strictly decreasing. Therefore we get for any  $y \in \partial S$  one of the following three cases:

1.  $y_i = N_i = \frac{1}{2p_i(N)}$ ,  $i = 1, 2$
2.  $y_1 > N_1 = z_1(N) > \frac{1}{2p_1(y)}$  and  $y_2 < N_2 = z_2(N) < \frac{1}{2p_2(y)}$
3.  $y_1 < N_1 = z_1(N) < \frac{1}{2p_1(y)}$  and  $y_2 > N_2 = z_2(N) > \frac{1}{2p_2(y)}$ .

This implies for any  $x \in S$  that for both  $i = 1, 2$  either  $x_i = N_i$  or  $\zeta_i(x) = z_i(y^i(x)) < N_i$ .

This shows that any  $x \in S_1 \times S_2$  results in a feasible payoff vector  $(\zeta_1(x), \zeta_2(x)) \leq N$  and that  $N_i$  is the unique optimal choice for player  $i$  independently of the other player's choice. Therefore  $N = (N_1, N_2)$  is the unique Nash equilibrium of  $\Gamma_S$  with dominant strategies  $N_i$ ,  $i = 1, 2$  for the two players. ♦



## 4. Concluding Remarks

What we have achieved in this paper is a direct one-stage implementation of the Nash bargaining solution in dominant strategies. Rather than giving a prescription for the organization of real bargaining our results shed some light on how Nash's solution is to be interpreted. As required in the Nash program it gives one specific non-cooperative foundation which appears quite different from those non-cooperative approaches in the literature. Of particular interest is the fact that our Nash equilibrium is unique and in dominant strategies.

The equal shares assumption in our economy  $\mathcal{E}_S$  which has its impact also on the payoff functions in  $\Gamma_S$  reflects Shapley's (1969) concept of "equity". Evaluating two players' utility levels by suitable prices defines endogenously transfer rates. In the equilibrium (i.e. the Nash solution) the rates of the two utilities used in the transition from 0 to N coincide with the transfer rates at N represented by  $p(N)$ .

The equilibrium "equilibrates" two opposite interests of the agents: On the one hand they want "their own" commodities to be expensive to induce the firm to produce a large quantity of it thereby creating a high profit. On the other hand they want it to be unexpensive to be able to buy a large amount of it. Any proposed efficient point  $y \in \partial Y$  could be sold at prices  $p(y)$ . Then half of the resulting revenue could be given to each of the two agents. If they could trade at prices  $p(y)$  in general this hypothetical market activity would improve exactly one agent. It would make the other one worse of because he could not buy back the quantity of his commodity he sold. Only the Nash solution N gives both agents their demand at  $p(N)$  without making use of any hypothetical market transactions. Only N leaves no hypothetical arbitrage possibilities. This is quite the same as with Shapley's  $\lambda$ -transfer value which is the unique one in a family of Shapley values of TU games associated with a given NTU game which can be realized without making use of transfer possibilities.

The results seem to suggest an interpretation of the Nash solution as an agreement which is forced by some kind of competitive pressure. It would be interesting to see our "hypothetical market opportunities" replaced by a real market structure in an explicit dynamical competitive model like for instance Gale (1986 a; b), Rubinstein & Wolinsky (1985), McLennan and Sonnenschein (1991).

An alternative way of looking at  $\mathcal{E}_S$  is at as a coalition production economy with a unit interval of players of two types (cf. Hildenbrand (1974)), where both types are represented by half of the interval. Any change of the weights of the two types would lead to an economy with different shares represented by  $\alpha, 1 - \alpha \in (0, 1)$ . These weights could be interpreted as bargaining power. The unique equilibrium of a representing economy  $\mathcal{E}_S^\alpha$  would be an asymmetric Nash solution with weights  $\alpha, 1 - \alpha$ .

Also  $\Gamma_S$  could be modified to  $\Gamma_S^\alpha$  in a straightforward way. Again the asymmetric Nash solution would turn out as the unique Nash equilibrium (in dominant strategies) of  $\Gamma_S^\alpha$ . Different bargaining powers would be represented by different budgets resulting in different payoff functions in  $\Gamma_S^\alpha$ .

Also an extension of these results to  $n > 2$  is straightforward. The dominant strategy equilibrium underlines the competitive aspect by the fact that every player can influence only his own payoff but is without any influence on the other players' payoffs.

The game proposed in the present paper might appear at first sight similar to a game where each player  $i$  proposes an amount of his utility and receives  $N_i$  if he proposes  $N_i$  and 0 otherwise. This is a way in which always implementation via equilibrium in dominant strategies could be established. But here the player  $i$  has to know  $N_i$ .

In the game  $\Gamma_S$ , however, the players need not know the concept of the Nash solution nor the specific point  $\varphi_N(S, 0) = N$ . They only have to be able to compare any efficient point with the resulting demand. This requires knowledge of  $(S, 0)$  but not of the point  $N = \varphi_N(S, 0)$  nor even of the concept of the Nash solution.

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